Доклады Болгарской академии наук
Comptes rendus de l'Académie bulgare des Sciences
Tome 32, N° 9, 1979

PHYSIQUE
Physique théorique

BACKLUND TRANSFORMATION IN THE CLASSICAL MASSIVE THIRRING MODEL

E. R. Nissimov, S. J. Pacheva

(Submitted by Academician I. T. Todorov on April 17, 1979)

A new geometric approach to nonlinear evolution equations with soliton solutions has been developed in the past few years [1–5]. The corresponding general Lie group framework was also presented [6]. It gives powerful methods for deriving (at least in principle) linear (Lax's form [7]) and more general nonlinear representations of soliton equations as integrability conditions for certain systems of first order partial differential equations. This fact turns out to have a natural geometric interpretation as the vanishing of a curvature form defined on a suitable fibre bundle [6].

In the present note our aim is to find in a systematic way a nonlinear representation and the associated Backlund transformation for the classical massive Thirring model (1). Let us recall that the complete integrability of that model has already been derived by means of the inverse scattering method [8]. As it will become clear below, the corresponding Lax form arises as a particular (degenerate) representation of a certain commutation algebra (6).

The classical massive Thirring model is a relativistically invariant theory of a complex c-number spinor field $\psi_a(x)$ $a=1, 2$ in two-dimensional space-time, specified by the Lagrangian:

$$L(x)=i/2\,\bar{\psi}\gamma^{\mu}\overleftrightarrow{\partial_{\mu}}\psi-m\bar{\psi}\psi-\lambda/4(\bar{\psi}\gamma^{\mu}\psi)^2,\quad \bar{\psi}=\psi^*\gamma^0. \tag{1}$$

Choosing a particular representation of the Dirac matrices:

$$\gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \gamma^1=\begin{pmatrix}0&-1\\1&0\end{pmatrix},$$

introducing light cone coordinates $\xi=1/2(x^0+x^1)$, $\eta=1/2(x^0-x^1)$ and removing the irrelevant mass m and coupling constant λ by a trivial rescaling of ξ, η and ψ, the equations of motion acquire the form:

$$i\partial_{\xi}\psi_1=\psi_2+\psi_2^*\psi_2\psi_1,\quad i\partial_{\eta}\psi_2=\psi_1+\psi_1^*\psi_1\psi_2. \tag{2}$$

Let us look for a (nonlinear) representation of eqs. (2):

$$\partial_{\xi}\Gamma_a=\mathcal{F}_a[\psi,\psi^*;\Gamma_b,\Gamma_b^*],\quad \partial_{\eta}\Gamma_a=\mathcal{G}_a[\psi,\psi^*;\Gamma_b,\Gamma_b^*] \tag{3}$$

$a, b=1,\ldots,N,$

Γ_a^* satisfying a system of the same kind with $\mathcal{F}_a^*[\ldots]=(\mathcal{F}_a[\ldots])^*$, $\mathcal{G}_a^*[\ldots]=(\mathcal{G}_a[\ldots])^*$. Accordingly, the integrability condition for (3) reads:

(4) $$\partial_\eta \mathcal{F}_a[\psi, \psi^*; \Gamma_b, \Gamma_b^*] = \partial_\xi \mathcal{G}_a[\psi, \psi^*; \Gamma_b, \Gamma_b^*].$$

Performing the differentiations in (4) and using (2), (3), we obtain after straightforward calculations:

$$\mathcal{F}_a[\psi, \psi^*; \Gamma_b, \Gamma_b^*] = i\psi_2 X_a^{(1)}(\Gamma_b, \Gamma_b^*) + i\psi_2^* X_a^{(2)}(\Gamma_b, \Gamma_b^*)$$
$$+ i\psi_2^*\psi_2 X_a^{(3)}(\Gamma_b, \Gamma_b^*) + iX_a^{(4)}(\Gamma_b, \Gamma_b^*),$$

(5) $$\mathcal{G}_a[\psi, \psi^*; \Gamma_b, \Gamma_b^*] = i\psi_1 Y_a^{(1)}(\Gamma_b, \Gamma_b^*) + i\psi_1^* Y_a^{(2)}(\Gamma_b, \Gamma_b^*)$$
$$+ i\psi^*\psi_1 Y_a^{(3)}(\Gamma_b, \Gamma_b^*) + iY_a^{(4)}(\Gamma_b, \Gamma_b^*).$$

The functions $X_a^{(i)}$, $Y_a^{(j)}$ satisfy the following commutation algebra:

(6) $$[X^{(i)}, Y^{(j)}] = \sum_{k=1}^{N} \{a^{ijk} X^{(k)} + b^{ijk} Y^{(k)}\},$$

$$a^{123} = b^{123} = a^{231} = b^{311} = a^{422} = 1,$$

$$a^{131} = b^{141} = a^{213} = b^{213} = b^{242} = b^{322} = a^{411} - 1,$$

$$[X^{(i)}, Y^{(j)}]_a \equiv \partial X_a^{(i)}/\partial \Gamma_b Y_b^{(j)} - \partial Y_a^{(j)}/\partial \Gamma_b X_b^{(i)},$$

where only the nonvanishing "structure constants" are written down. As it stands (6) does not form a closed Lie algebra, but it is not difficult to find some particular degenerate representations which will be of definite importance. First we present a one-dimensional (N=1) nonlinear representation of (6):

(7)
$$X^{(1)} = \varkappa, \quad X^{(2)} = -\varkappa\Gamma^2, \quad X^{(3)} = -\Gamma, \quad X^{(4)} = \varkappa^2\Gamma;$$
$$Y^{(1)} = \varkappa^{-1}, \quad Y^{(2)} = -\varkappa^{-1}\Gamma^2, \quad Y^{(3)} = -\Gamma, \quad Y^{(4)} = \varkappa^{-2}\Gamma,$$

where $\varkappa$ is an arbitrary complex number. Substituting (7) into (5) and (3), we obtain:

(8)
$$\partial_\xi \Gamma = i(\varkappa^2 - \psi_2^*\psi_2)\Gamma + i\varkappa(\psi_2 - \psi_2^*\Gamma^2)$$
$$\partial_\eta \Gamma = i(\varkappa^{-2} - \psi_1^*\psi_1)\Gamma + i\varkappa^{-1}(\psi_1 - \psi_1^*\Gamma^2)$$

and analogous equations for Γ^*.

We can also find another two-dimensional (N=2) linear representation of (6):

$$X_a^{(i)} = \sum_{b=1}^{2} x_{ab}^{(i)}\Gamma_b, \quad Y_a^{(j)} = \sum_{b=1}^{2} y_{ab}^{(j)}\Gamma_b;$$

(9) $$x^{(1)} = \varkappa/2(\sigma_1 - i\sigma_2), \ x^{(2)} = \varkappa/2(\sigma_1 + i\sigma_2), \ x^{(3)} = 1/2\sigma_3, \ x^{(4)} = -\varkappa^2/2\sigma_3;$$

$$y^{(1)} = 1/2\varkappa(\sigma_1 - i\sigma_2), \ y^{(2)} = 1/2\varkappa(\sigma_1 + i\sigma_2), \ y^{(3)} = 1/2\sigma_3, \ y^{(4)} = -1/2\varkappa^2\sigma_3,$$

where σ_i ($i=1, 2, 3$) are the Pauli matrices. Substituting (9) into (5) and (3), one immediately recognizes the equations of the linear scattering problem associated with (2) [8]:

$$\hat{X}_\xi \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} = 0, \quad \hat{X}_\eta \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} = 0,$$

(10) $$\widehat{X}_\xi = i\partial_\xi + 1/2(\psi_2^*\psi_2 - \varkappa^2)\sigma_3 + \varkappa\begin{pmatrix} 0 & \psi_2^* \\ \psi_2 & 0 \end{pmatrix}$$

$\widehat{X}_\eta$ is obtained from $\widehat{X}_\xi$ by replacing $\varkappa \to \varkappa^{-1}$, $\xi \to \eta$, $1 \to 2$.

Now the following geometric interpretation of eqs (8) and (10) can be given. We rewrite (8) and (10) in the form:

$$\partial_\mu \Gamma_a(x) = \sum_{k=1}^{3} A_\mu^k(x) T_{k,a}(\Gamma)$$

(11) $$A_\xi^1(x) = -\varkappa(\psi_2 + \psi_2^*),\ A_\xi^2(x) = i\varkappa(\psi_2 - \psi_2^*),\ A_\xi^3(x) = \varkappa^2 - \psi_2^*\psi_2,$$
$$A_\eta^1(x) = -\varkappa^{-1}(\psi_1 + \psi_1^*),\ A_\eta^2(x) = i\varkappa^{-1}(\psi_1 - \psi_1^*),\ A_\eta^3(x) = \varkappa^{-2} - \psi_1^*\psi_1.$$

Here $T_k(\Gamma)$ are generator functions of a bilocal [6] SU (2)-group action on the corresponding representation space resp.:

$$T_1(\Gamma) = i/2(\Gamma^2 - 1),\quad T_2(\Gamma) = \Gamma 1/2(^2 + 1),\quad T_3(\Gamma) = i\Gamma;$$

$$T_{k,a}(\Gamma) = \sum_{b=1}^{2}[-i/2(\sigma_k)^{ab}]\Gamma_b,\quad a = 1,\ 2;\quad k = 1,\ 2,\ 3.$$

In the notations of (11) the equations of motion (2) can be cast into the form: $F_{\mu\nu}^k \equiv \partial_\mu A_\nu^k - \partial_\nu A_\mu^k - \varepsilon^{klm} A_\mu^l A_\nu^m = 0$, i. e. $A_\mu^k(x)$ in the parametrization of (11) defines a flat connection in a fibre bundle with the space — time as a base, certain N-dimensional manifold as fibres and structure group SU(2), acting (in general nonlinear) on the cross sections of this bundle given by means of the functions $\Gamma_a(x)$.

Let us now turn to the construction of the Backlund transformation for (2). The latter should have the following general form:

$$\widetilde{\psi}_\alpha = B_\alpha[\psi_\alpha, \psi_\alpha^*; \Gamma, \Gamma^*],\quad \alpha = 1,\ 2;$$

(12) $$i\partial_\xi B_1 = B_2 + |B_2|^2 B_1,\quad i\partial_\eta B_2 = B_1 + |B_1|^2 B_2$$

and analogously for $\widetilde{\psi}_\alpha^*$. Here $\Gamma(x)$ satisfies eqs (8). One can easily find the concrete structure of $B_\alpha[\ldots]$ by means of straightforward manipulations with (12), (2), (8) and accounting for the U(1) internal symmetry of (2), (8):

(13) $$B_\alpha[\psi_\alpha, \psi_\alpha^*; \Gamma, \Gamma^*] = \psi_\alpha C_\alpha(u) + \Gamma D_\alpha(u),\quad \alpha = 1,\ 2,$$

where the functions C_α, D_α depend on a single variable $u = \Gamma\Gamma^*$. Substituting (13) into (12) and using (2), (8), we get an overdetermined system of 14 equations for the 4 unknown functions $C_\alpha(u)$, $D_\alpha(u)$:

$$C_1C_2D_2^* = -(\varkappa^* u + \varkappa)dG_1/du,\quad C_1C_2^*D_2 = (\varkappa u + \varkappa^*)dC_1/du,$$
$$C_2 - C_1 + \varkappa D_1 + uC_2D_1D_2^* + u(\varkappa + u\varkappa^*)dD_1/du = 0,$$
$$(\varkappa u + \varkappa^*)dD_1/du + \varkappa D_1 - C_2^*D_1D_2 = 0,\quad |C_2(u)| = 1,$$
(14) $$(\varkappa^2 - \varkappa^{*2})dC_1/du + C_1|D_2|^2 = 0,$$
$$\varkappa^2 D_1 + (\varkappa^2 - \varkappa^{*2})udD_1/du + uD_1|D_2|^2 + D_2 = 0,$$

plus 7 further equations which are obtained from (14) by the substitutions $1 \rightleftarrows 2$, $\varkappa \rightleftarrows \varkappa^{-1}$. From the first and the sixth eqs (14) and from the analogous ones in the second subsystem one gets:

(15) $$C_2=\frac{\varkappa^*u+\varkappa}{\varkappa^2-\varkappa^{*2}}D_2,\quad C_1=\frac{(\varkappa^*)^{-1}u+\varkappa^{-1}}{\varkappa^{-2}-(\varkappa^*)^{-2}}D_1.$$

From the second, fifth and the fourth eqs (14) and the analogous ones in the second subsystem and from (15) it follows that:

$$d/du\ln C_1=\frac{\varkappa^2-\varkappa^{*2}}{|\varkappa^*u+\varkappa|^2},\quad d/du\ln C_2=\frac{\varkappa^{-2}-(\varkappa^*)^{-2}}{|(\varkappa^*)^{-1}u+\varkappa^{-1}|^2},$$

(16) $$d/du\ln D_1=-(u+\varkappa/\varkappa^*)^{-1},\quad d/du\ln D_2=-(u+\varkappa^*/\varkappa)^{-1}.$$

The general solution of (16) contains some arbitrary constants which have to be fixed as appropriate functions of $\varkappa$, $\varkappa^*$, such that the whole overdetermined system is satisfied.

Thus we arrive at our main result — the Backlund transformation for (1):

(17) $$\widetilde{\psi}_1=\left[\psi_1+\frac{\varkappa^{-2}-(\varkappa^*)^{-2}}{(\varkappa^*)^{-1}\Gamma\Gamma^*+\varkappa^{-1}}\Gamma\right]\frac{\varkappa\Gamma\Gamma^*+\varkappa^*}{\varkappa^*\Gamma\Gamma^*+\varkappa}$$

and analogously for $\widetilde{\psi}_2$ by replacing $1\to2$, $\varkappa\to\varkappa^{-1}$.

Finally, let us note that with the aid of the simple pseudopotential Γ (8) one can write down the infinite number of conservation laws in the model (1) compactly as:

(18) $$\partial_\xi(\varkappa^{-1}\psi_1^*\Gamma-(\varkappa^*)^{-1}\Gamma^*\psi_1)=\partial_\eta(\varkappa\psi_2^*\Gamma-\varkappa^*\Gamma^*\psi_2)$$

or equivalently after substituting the power series expansion for Γ:

$$\Gamma=\sum_{n=0}^{\infty}\varkappa^{2n+1}\Gamma_n,$$

(18′) $$\partial_\xi(\psi_1^*\Gamma_n-\Gamma_n^*\psi_1)=\partial_\eta(\psi_2^*\Gamma_{n-1}-\Gamma_{n-1}^*\psi_2);$$

(19) $$\Gamma_{n+1}=-i\partial_\eta\Gamma_n+\psi_1^*\psi_1\Gamma_n+\psi_1^*\sum_{k+l=n}\Gamma_k\Gamma_l,\quad \Gamma_0=-\psi_1.$$

Formulas (17)—(19) may be compared with their analogues in the anticommuting classical massive Thirring model [9, 10].

Institute of Nuclear Research
and Nuclear Energy
Bulgarian Academy of Sciences,
Sofia 1113, Bulgaria

Steklov Mathematical Institute
USSR Academy of Sciences
Leningrad 191011, USSR

REFERENCES

[1] H. D. Wahlquist, F. B. Estabrook. J. Math. Phys. 16, 1975, 1 [2] J. Corones. Ibid. 17, 1976, 756 [3] R. Hermann. Phys. Rev. Lett. 36, 1976, 835. [4] H. C. Morris. J. Math. Phys. 18, 1977, 533 [5] M. Crampin, F. Pirani, D. Robinson. Lett. Math. Phys. 2, 1977, 1. [6] J. Corones, B. L. Markovsky, V. A. Rizov. J. Math. Phys. 18, 1977, 2207. [7] P. D. Lax. Comm. Pure Appl. Math. 21, 1968, 467. [8] E. A. Kuznetsov, A. V. Michailov. Theor. Math. Phys. 30, 1977, 303. [9] P. P. Kulish, E. R. Nissimov. Ibid. 29, 1976, 161; [10] A. Izergin, J. Stehr. DESY preprint 76/60, 1976.